Robust Network Connectivity: a Percolation Study

Arman Mohseni Kabir
Dept. of Physics
University of Massachusetts
Amherst, MA, USA
arman@physics.umass.edu

Mihir Pant
Dept. of EECS
MIT
Cambridge MA, USA
mpant@mit.edu

Don Towsley
CICS
University of Massachusetts
Amherst, MA, USA
towsley@cs.umass.edu

Saikat Guha
College of Optical Sciences
University of Arizona
Tucson AZ, USA
saikat@optics.arizona.edu

Paul Yu
Army Research Lab
Adelphi MD, USA
paul.l.yu.civ@mail.mil

Ananthram Swami
Army Research Lab
Adelphi MD, USA
ananthram.swami.civ@mail.mi

Communication networks, power grids, and transportation networks are all examples of networks whose performance depends on reliable connectivity of their underlying network components even in the presence of usual network dynamics due to mobility, node or edge failures, and varying traffic loads. Percolation theory quantifies the threshold value of a local control parameter—such as a node occupation (resp., deletion) probability q or an edge activation (resp., removal) probability p—above (resp., below) which there exists a *giant connected component* (GCC), a connected component comprising of occupied nodes and active edges whose size is proportional to the size of the network itself. Any pair of nodes in the GCC is connected via at least one path. The mere existence of the GCC itself does not guarantee robustness, e.g., to network dynamics. In this paper, we explore new percolation thresholds that guarantee not only spanning network connectivity, but also robustness to failures.

We analyze four measures of robust connectivity, explore their interrelationships, and numerically evaluate our results on the robust percolation thresholds for the 2D square lattice network as an example:

(1) ***k*-strong-connectivity:** The subset of the network nodes in which every pair of nodes has at least k-edge-disjoint paths connecting them. This means that removal of *k*-1 edges from the k-strong-component will not disconnect the nodes in the component. As we will see in the results section of the paper, this is a very strong measure of robustness.

(2) ***k*-connectivity:** The subset of the network nodes in which each pair of nodes has at least k-edge-disjoint paths connecting them except that the path may contain nodes that act as conduits connecting two other nodes and are not themselves a part of the component. Newman *et al* (1). showed that the percolation thresholds for any configuration model network are the same as regular percolation (with no additional demand of robustness), although the absolute size of the GCC in the supercritical regimes varies with k.

(3) ***k*-shell:** The subset of the network nodes in which each node has at least k nearest neighbors each of which is also in the *k*-core component. The concept of *k*-core and decomposing a complex network to its *k*-core components was recently applied to many real-world networks (the Internet, the WWW, cellular networks, etc.) (2).

(4) ***k*-core:** The subset of the network nodes in which each node in the component must have at least k nearest neighbors.

An important property of these measures of robustness is that they are nested, that is, any *k*-component in any of the four models is a subset of the $(k-1)^{th}$ component. As its shown in Fig. 1(a) there is also a hierarchical relation between these measures of robustness, *k*-strong-connectivity being the strongest one followed by *k*-core and *k*-connectivity. There is no established hierarchical relationship between *k*-core and *k*-connectivity since they belong to distinct measures of robustness as they can have different robustness properties in different networks. The next measures in the hierarchy are *k*-shell and regular connectivity which have very similar robustness properties.

Our theoretical results are based on theorems developed by Grimmet (3), which gives the disjoint path properties of the regular percolation in the square grid above the percolation threshold. In short, the theorem states that w.h.p there are order N disjoint right-left and up-down path crossings of a box of sides N in the bond percolation problem on square grid for occupation probabilities above the percolation threshold ($p > p_c$). This disjoint highway structure implies the existence of a renormalized square grid with order N^2 number of nodes. The existence of this renormalized square grid can then be used to establish the *k*-connectivity properties of the square grid above the percolation threshold.

One important aspect of percolation theory should be considered when using it for practical applications. Percolation theory works in the limit, when the size of the network goes to infinity. However, practical applications (i.e. communication networks), deal with much smaller sized networks. This can be dealt with by using the well-established theory of finite size scaling for phase transitions. Using this approach, we can get good approximations of percolation theory in the finite-size limit of the network.

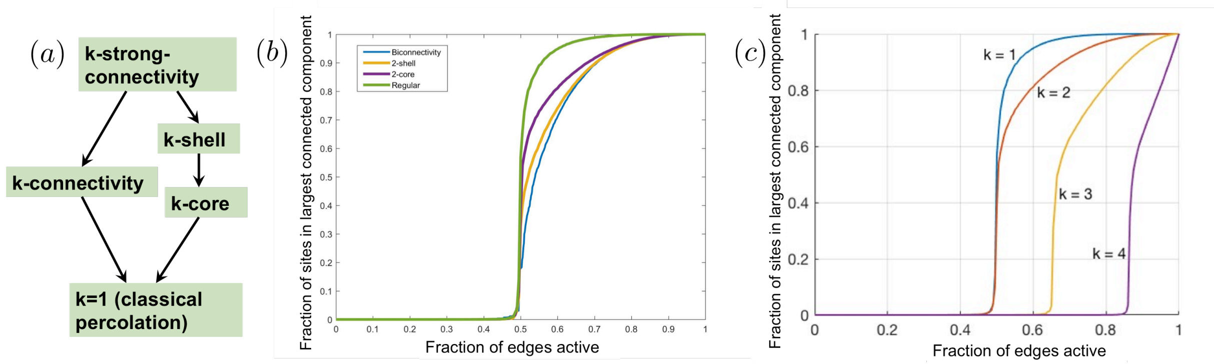

Fig. 1. (a) Stronger measure of robustness points to progressively weaker ones; (b) (Bond) percolation curves for all measures of robustness we consider in this paper for their respective $k = 2$ versions – showing that the percolation thresholds are the same for $k = 2$ for all the measures but the size of the GCC are different in the supercritical (with the relative order agreeing with the relative strengths of the measures of robustness; (c) (Bond) percolation thresholds for the k-core measure with $k = 1, \dots, 4$. All results are for the 2D square lattice.

The algorithm for finding the percolation behavior of our model is similar in spirit to the fast percolation algorithm of Newman and Ziff (4) in which we start with an empty graph and randomly occupy bonds one by one until we occupy all the bonds in the square grid. Our simulation results are based on implementations of algorithms developed by Hopcroft *et al.* (5) and Gutwenger *et al.* (6) which uses depth first search trees and SPQR trees to find the k-components of a given instant of our network.

We can argue that for $k = 2$, for any graph, the percolation threshold is the same for all four robustness measures and equals the regular percolation threshold. But, the size of the GCC in the supercritical regime differs (see Fig. 1(b)). We also show for the square grid, that the percolation threshold for 3-connectivity is the same as that for regular percolation. In addition, we show that for bond percolation in the square grid there is no 3-core and 3-strong-component if the square grid does not have periodic boundary conditions and if the lattice has periodic boundary conditions the percolation threshold is the same for both models. We further conjecture that the percolation threshold for 4-connectivity is the same as the threshold for the regular connectivity (which is also equal to 2-connectivity and 3-connectivity).

We numerically evaluated various percolation thresholds for all these measures for the square grid. Fig. 1(c) shows the percolation plots for the k-core measure with $k = 1, \dots, 4$. We were able to argue why the threshold for $k = 4$ should be strictly less than 1. In ongoing work, we are translating these connectivity-based network robustness measures to understand how to design and control a software defined wireless network to realize distributed analytics that are robust to network dynamics.

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